Application Of The Constant Boundary Element Method To The Two Group-Two Dimensional Neutron Diffusion Equation

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Abstract

A boundary integral equation (BIE) is developed for the application of the boundary element method to the two group neutron diffusion equations. Because the scattering effects are accounted by redefining the unknowns, BIE include no explicit scattering term. Constant boundary elements are utilized for spatial discretization.

Keywords: Neutron diffusion equation, boundary integral equation, Boundary Element Method

Özet

Bu çalışmada öncelikle, iki gruplu iki boyutlu nötron difüzyon denklemlerine sınır elemanları yöntemini uygulamak amacıyla, nötron difüzyon denklemlerinin eşdeğeri sınır integral denklemleri türetilmiştir. Sınır integral denklemleri türetilirken, nötron difüzyon denkleminde sözkonusu olan saçılma etkisi elimine edilmiştir. Bu nedenle bilinmeyenler arasında saçılma terimi yer almamaktadır.

Elde edilen sınır integral denklemleri üzerinde uzaysal ayrıklaştırma için Sabit Elemanlı Sınır Elemanları Yöntemi kullanılmıştır.

Anahtar Kelimeler: Nötron difüzyon denklemi, sınır integral denklemi, Sınır Elemanları Yöntemi

1. Inroduction

The existence of fission, external and slowing down source require internal meshes in the application of the Boundary Element Method (BEM) to the neutron diffusion equation.

Since in one-group diffusion theory slowing down volume integrals do not exist, in the previous works on one-group neutron diffusion researchers have been concerned on transforming one-group fission source and external source volume integrals into equivalent surface integrals by using the multiple reciprocity method [1], [2]. In multigroup diffusion theory, group slowing down source volume integrals are added to the boundary integral equation. In the case of no fission and within the context of two or three group theory on the transformation of the slowing down source volume integrals into equivalent surface integral has been worked out [3]. Here we will discuss the same transformation in the presence of fission.

2. The derivation of the boundary integral equation

In diffusion theory, time independent two-group diffusion equations for a single region could be written as:

$$\nabla^2\Phi_g(\vec{r}) - k_g^2\,\Phi_g(\vec{r}) = -\frac{S_g(\vec{r})}{D_g} \qquad g=1,2\ . \tag{1}$$

And in this case, the group source term without upscattering is:

$$S_g(\vec{r}) = \begin{cases} q_1(\vec{r}) & g=1 \\ q_2(\vec{r}) + \Sigma_{s2\leftarrow 1}\,\Phi_1(\vec{r}) & g=2 \end{cases} \tag{2}$$

where;

- k_g : group inverse diffusion length
- D_g : group diffusion constant
- $\Sigma_{s,g2\leftarrow 1}$: scattering cross section from group 1 to group 2
- $S_g(\vec{r})$: group source term
- $q_g(\vec{r})$: group fission and/or external source.

For a homogeneous region V, surface S is seperated as S_v and S_r ($S=S_r+S_v$). Assume that, on S_v the vacuum boundary condition is valid and on S_r the reflection boundary condition prevails. The group infinite medium Green's function is defined as the solution of

$$\nabla^2 G_g(\vec{r},\vec{\rho}) - k_g^2\, G_g(\vec{r},\vec{\rho}) = -\delta(\vec{r}-\vec{\rho})\,. \tag{3}$$

Multiplying eqn.(1) by the fundamental solution, integrating over V and using Green's Second Identity

$$C(\vec{\rho})\,\Phi_g(\vec{\rho}) + \int_{S_r} \frac{\partial G_g(\vec{r},\vec{\rho})}{\partial n}\,\Phi_g(\vec{r})\,dS - \int_{S_v} \frac{\partial \Phi_g(\vec{r})}{\partial n}\,G_g(\vec{r},\vec{\rho})\,dS = \int_V \frac{q_g(\vec{r})}{D_g}\,G_g(\vec{r},\vec{\rho})\,dS + S_g(\vec{\rho}) \tag{4}$$

is obtained, where

$$S_g(\vec{\rho}) = \begin{cases} 0 & g=1 \\ \dfrac{\Sigma_{s2\leftarrow 1}}{D_2} \int\limits_V G_2(\vec{r},\vec{\rho})\,\Phi_1(\vec{r})\,dV & g=2 \end{cases} \tag{5}$$

and

$$C(\vec{\rho}) = \begin{cases} 1 & \vec{r} \in V / S \\ \dfrac{\vartheta}{2\pi} & \vec{r} \in S \end{cases} \tag{6}$$

ϑ is the internal angle in radians at the boundary point with the position vector $\vec{\rho}$.

In the criticality eigenvalue problems, the group fission source term is:

$$q(\vec{r}) = \frac{\chi_g}{k_{eff}} f(\vec{r})\,; \qquad \text{where} \qquad f(\vec{r}) = \sum_{l=1}^{2} \nu\Sigma_{f,l}(\vec{r})\Phi_l(\vec{r})\,. \tag{7}$$

If eqn.(3) is rewritten for g=2

$$\begin{aligned} &\nabla^2 G_2(\vec{r},\vec{\rho}) - k_2^2\, G_2(\vec{r},\vec{\rho}) = -\delta(\vec{r}-\vec{\rho}) \\ &\qquad G_2(\vec{r},\vec{\rho}) = \frac{1}{k_2^2}\left[\nabla^2 G_2(\vec{r},\vec{\rho}) + \delta(\vec{r}-\vec{\rho})\right] \end{aligned} \tag{8}$$

and when this is incorporated into eqn.(5)

$$\begin{aligned} S_2(\vec{\rho}) &= \frac{\Sigma_{s2\leftarrow 1}}{D_2}\frac{1}{k_2^2}\int\limits_V \left[\nabla^2 G_2(\vec{r},\vec{\rho}) + \delta(\vec{r}-\vec{\rho})\right]\Phi_1(\vec{r})\,dV \\ &= \frac{\Sigma_{s2\leftarrow 1}}{D_2}\frac{1}{k_2^2}\left[C(\vec{\rho})\Phi_1(\vec{\rho}) + \int\limits_V \Phi_1(\vec{r})\nabla^2 G_2(\vec{r},\vec{\rho})\,dV\right] \end{aligned} \tag{9}$$

is obtained. Using Green's Second Identity to the second term of right-hand-side of eqn.(9), it becomes

$$\begin{aligned} \int\limits_V \Phi_1(\vec{r})\nabla^2 G_2(\vec{r},\vec{\rho})\,dV &= \int\limits_V G_2(\vec{r},\vec{\rho})\nabla^2\Phi_1(\vec{r})\,dV \\ &+ \int\limits_S \Phi_1(\vec{r})\frac{\partial G_2(\vec{r},\vec{\rho})}{\partial n}\,dS - \int\limits_S G_2(\vec{r},\vec{\rho})\frac{\partial \Phi_1(\vec{r})}{\partial n}\,dS. \end{aligned} \tag{10}$$

Eqn.(1) is rewritten for g=1 and result is inserted into eqn.(10) one gets:

$$\begin{aligned} &\nabla^2\Phi_1(\vec{r}) = k_1^2\,\Phi_1(\vec{r}) - \frac{q_1(\vec{r})}{D_1} \\ &\int\limits_V \Phi_1(\vec{r})\nabla^2 G_2(\vec{r},\vec{\rho})\,dV = \int\limits_V k_1^2 G_2(\vec{r},\vec{\rho})\Phi_1(\vec{r})\,dV - \int\limits_V \frac{q_1(\vec{r})}{D_1} G_2(\vec{r},\vec{\rho})\,dV \\ &\qquad + \int\limits_S \Phi_1(\vec{r})\frac{\partial G_2(\vec{r},\vec{\rho})}{\partial n}\,dS - \int\limits_S G_2(\vec{r},\vec{\rho})\frac{\partial \Phi_1(\vec{r})}{\partial n}\,dS. \end{aligned} \tag{11}$$

Using eqn.(11), eqn.(9) can be organized as:

$$S_2(\vec{\rho}) = \frac{\Sigma_{s2\leftarrow 1}}{D_2}\frac{1}{\left(k_2^2-k_1^2\right)}C(\vec{\rho})\,\Phi_1(\vec{\rho}) - \frac{\Sigma_{s2\leftarrow 1}}{D_2}\frac{1}{\left(k_2^2-k_1^2\right)}\frac{1}{D_1}\left[\int_V G_2(\vec{r},\vec{\rho})\,q_1(\vec{r})\,dV\right] + \frac{\Sigma_{s2\leftarrow 1}}{D_2}\frac{1}{\left(k_2^2-k_1^2\right)}\left[\int_S \Phi_1(\vec{r})\frac{\partial G_2(\vec{r},\vec{\rho})}{\partial n}dS - \int_S G_2(\vec{r},\vec{\rho})\frac{\partial \Phi_1(\vec{r})}{\partial n}dS\right] \quad (12)$$

By defining

$$s_{12} = \frac{\Sigma_{s2\leftarrow 1}}{D_2}\frac{1}{\left(k_2^2-k_1^2\right)}$$

$$\delta_g(\vec{r}) = \begin{cases}\Phi_1(\vec{r}) & g=1\\ \Phi_2(\vec{r}) - s_{12}\Phi_1(\vec{r}) & g=2\end{cases} \quad ; \quad z_g(\vec{r}) = \begin{cases}\dfrac{q_1(\vec{r})}{D_1} & g=1\\ \dfrac{q_2(\vec{r})}{D_2} - s_{12}\dfrac{q_1(\vec{r})}{D_1} & g=2\end{cases} \quad (13)$$

and substituting eqn (12) into group integral eqn (4) we obtain:

$$C(\vec{\rho})\,\delta_g(\vec{\rho}) + \int_{S_r}\frac{\partial G_g(\vec{r},\vec{\rho})}{\partial n}\delta_g(\vec{r})\,dS - \int_{S_v}\frac{\partial \delta_g(\vec{r})}{\partial n}G_g(\vec{r},\vec{\rho})\,dS = \int_V z_g(\vec{r})\,G_g(\vec{r},\vec{\rho})\,dV\,. \quad (14)$$

3. Boundary element discretization

We assume that the mentioned two-dimensional homogeneous region is segmented into N boundary elements which is of the constant type. For constant boundary elements, the center of each element is a node. Also, volume V is seperated M sub-volumes. Taking as $\vec{\rho}$ the position vector of a node, eqn.(14) can be written as a boundary integral equation:

$$C_i\,\delta_g(\vec{\rho}_i) + \sum_{\substack{j=1\\ j\in S_r}}^{N}\int_{S_j}\frac{\partial G_g(\vec{r},\vec{\rho}_i)}{\partial n}\delta_g(\vec{r}_j)\,dS - \sum_{\substack{j=1\\ j\in S_v}}^{N}\int_{S_j}G_g(\vec{r},\vec{\rho}_i)\frac{\partial \delta_g(\vec{r}_j)}{\partial n}dS = \sum_{k=1}^{M}\int_{V_k} z_g(\vec{r})\,G_g(\vec{r},\vec{\rho}_i)\,dV \quad (15)$$

and

$$h_{ij}^{g}(\vec{\rho}_i) = C_i\,\delta_g(\vec{\rho}_i) + \int_{S_j}\frac{\partial G_g(\vec{r},\vec{\rho}_i)}{\partial n}\,dS \qquad \vec{r}_j \in S_r$$

$$g_{ij}^{g}(\vec{\rho}_i) = \int_{S_j} G_g(\vec{r},\vec{\rho}_i)\,dS \qquad \vec{r}_j \in S_v$$

$$\xi_j^g = \frac{\partial \delta_g(\vec{r}_j)}{\partial n} \qquad \vec{r}_j \in S_v$$

$$\delta_j^g = \delta_g(\vec{r}_j) \qquad \vec{r}_j \in S_r$$

$$f_i^g(\vec{\rho}_i) = \sum_{k=1}^{M} \int_{V_k} G_g(\vec{r}, \vec{\rho}_i)\, z_g(\vec{r}) dS \qquad \vec{r}_j \in V. \tag{16}$$

With matrix notation, eqn.(15) is written compactly as:

$$\underline{\underline{H}}\,\underline{\delta} - \underline{\underline{G}}\,\underline{\xi} = \underline{F}\,. \tag{17}$$

In this equation, H and G are NxN matrices; ξ, δ and F are vectors whose dimension N. Since either flux or current is zero, because of boundary conditions, there are actually N unknown in eqn.(17). When, all unknowns is collected on same side eqn.(17) can be rewritten as:

$$\underline{\underline{A}}\,\underline{u} = \underline{F}\,. \tag{18}$$

4. Numerical results

The developed formulation is implemented in FORTRAN program BEMG2. BEMG2 is capable of handling both fixed source and criticality problems for a maximum of two groups.

The results obtained with BEMG2 have been compared with known analytical solutions and the results of other computer programs (BEMFS and FEND[4]). Thus, BEMG2 is validated.

The data of results presented in here are found in reference [5].

For infinite medium problems, the medium has been assumed to be a square with side length 2 cm and reflected boundary condition is applied on all sides.

Table 1. Infinite Medium One Group Fission Source Problem's Results

	Analytical	Numerical			
		4 nodes	Error	32 nodes	Error
k_∞	1	1.1345	%13.45	1.0017	%0.17
$\phi_{(inner)}$	3.12×10^{13}	3.1209×10^{13}	%0.03	3.1264×10^{13}	%0.21
$\phi_{(boundary)}$	-	0.2500	-	0.2504	-

Table 2. Infinite Medium Two Group Fission Source Problem's Results

	Analytical	Numerical			
		4 nodes	Error	32 nodes	Error
k_∞	1.8108	1.8337	%1.26	1.8111	%0.02
$\phi_{1(inner)}$	$1.8157\text{x}10^{14}$	$1.82\text{x}10^{14}$	%0.24	$1.82\text{x}10^{14}$	%0.24
$\phi_{2(inner)}$	$1.0162\text{x}10^{15}$	$1.02\text{x}10^{15}$	%0.37	$1.02\text{x}10^{15}$	%0.37
$\phi_{1(boundary)}$	-	1.4586	-	1.4546	-
$\phi_{2(boundary)}$	-	8.1363	-	8.1408	-

System with side length 50 cm is considered for fission source problem. But the problem has been solved using only the upper-right octant, because of symmetry. This system can be seen in Figure 1. In this figure, V denotes the vacuum boundary condition and R indicates the reflecting boundary condition.

Figure 1. Octant For Bare, Homogeneous Medium One Group Fission Source Problem

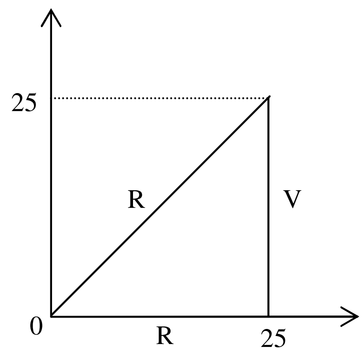

Table 3. Bare, Homogeneous Medium One Group Fission Source Problem's Results

	Analytical	Numerical			
		12 nodes	Error	24 nodes	Error
k_{eff}	0.9230	0.8931	%3.24	0.9177	%0.57

Table 4. Bare, Homogeneous Medium Two Group Fission Source Problem's Results

	Analytical	Numerical			
		12 nodes	Error	24 nodes	Error
k_{eff}	1.9652	1.9789	%0.70	1.9662	%0.05

5. Conclusion

In this work, the results obtained with BEMG2 have been compared with known analytical solutions and the results of other computer programs.

It has been observed that when the number of nodes on boundary increases, the accuracy of program increases validating the proposed formulation.

6. References

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