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Application Of The Adomian Decomposition Method To The One Group Neutron Diffusion Equation

Şükran Çavdar

cavdar@itu.edu.tr / Energy Institute / Istanbul Technical University

ABSTRACT – In this work, we consider two dimensional one group neutron diffusion equations for multiplying media and solve it through the Adomian Decomposition Method (ADM) in order to exploit its merit of avoiding assumptions on the original problem and achieving a solution through an iterative computational scheme. There is a vast amount of literature on mathematical methods for solving linear or nonlinear ordinary or partial differential equations, however, in order to apply these methods to problems arising in science and engineering, usually it is inevitable to make modifications to the original problem to have a certain form required by the particular method. Moreover, in most cases, a high amount of computational power is required. The ADM, proposed by Adomian and modified by Wazwaz, has been proved useful in obtaining closed form or numerical approximations for the solutions of many such problems involved with algebraic, linear/non-linear, ordinary/partial differential equations, integro-differential, integral or differential delay equations while making it possible to avoid linearization and modifications to the original problem which could correspond to unrealistic assumptions. Besides, the resulting computational schemes are efficient with high accuracy and generally, a rapidly convergent series solution is achieved. Being motivated with these facts, in this work, we apply the ADM to solve one group neutron diffusion equations. We present both analytical and numerical results.

Keywords: *adomian decomposition method, neutron diffusion equation.*

I. INTRODUCTION

There is a vast amount of literature on mathematical methods for solving differential equations that have linear or nonlinear, ordinary or partial nature. However, in order to apply these methods to problems arising in science and engineering [1-18], usually it is inevitable to make modifications to the original problem to have a certain form required by the particular method. Moreover, in most cases, numerical results require a high amount of computational power. The Adomian Decomposition Method (ADM), proposed by Adomian [1-14], has been proved useful in obtaining solutions for many such problems involving algebraic, linear/non-linear, ordinary/partial differential equations, integro-differential, integral or differential delay equations while making it possible to avoid linearizations and modifications to the original problem which could correspond to unrealistic assumptions to hold. The solution is obtained in a series form in which the terms are computed in an iterative manner. A partial sum corresponds to an approximate solution and through numerical evaluation of the terms, a numerical approximation to the solution is obtained. It is often the case that the series exhibit rapid convergence, the computational schemes that arise are relatively convenient that avoid complicated algebraic manipulations and numerical evaluation of them yield relatively efficient results with high accuracy [12], [14]. In a recent work, we have applied the ADM to fixed source neutron diffusion equations and achieved the analytic result provided by the traditional methods [18] recursively in a straightforward manner. In this work, we consider a multiplying media scenario for a two dimensional-one group system and exploit the aforementioned merits of the ADM for the solution of neutron diffusion equation considering a one group problem.

II. THE ADM FOR SOLVING DIFFERENTIAL EQUATIONS

The ADM relies on the fact that it is possible to decompose the solution of equations involving linear and/or nonlinear operators F_i given by

$$F_i\left[u_i(\vec{x})\right] = g_i(\vec{x}), \quad i = 1,..,l \tag{1}$$

for i = 1, 2, ..., l where $\vec{x}$ is an independent variable, $g_i(\vec{x})$ is a known function and $\{u_i(\vec{x})\}_{i=1,2,...l}$ is referred to as the solution [1] in series form as

$$u(\vec{x}) = \sum_{i=0}^{\infty} u_i(\vec{x}) \tag{2}$$

where the corresponding n-term partial sum is given by

$$\varphi_n(\vec{x}) = \sum_{i=0}^{N} u_i(\vec{x}) \tag{3}$$

We note that

$$\lim_{n\to\infty} \varphi_n(x) = u(x) \tag{4}$$

and the convergence is rapid [12], [14]. Moreover, the n-term partial sum is an approximate solution. Hence, the computation of the solution is equivalent to the computation of $u_i(\vec{x})$ for i=0, 1, 2, ...n-1. ADM proposes a recursive formula for finding these terms, starting with a decomposition of the general differential operator F that represents a general nonlinear ordinary differential operator that bears both linear and nonlinear terms as

$$F = L + R + N \tag{5}$$

where L+R and N are the linear and non-linear parts respectively. Here, L is the highest order derivative part which is invertible and R is the remaining part of the linear operator. Thefore (1) can be written as

$$Lu + Ru + Nu = g \tag{6}$$

Solving this equation for Lu yields

$$Lu = g - Ru - Nu \tag{7}$$

Since L is invertible, after multipliying both sides of the equation with L^{-1}, we end up with

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{8}$$

If the original problem given by (1) is an intitial-value problem, it is possible to treat the integral operator L^{-1} as definite integrations from 0 to x. If L is a second order operator, L^{-1} is a two-fold integration operator and $L^{-1}Lu = u - u(0) - xu'(0)$. For boundary value problems, indefinite integrations are used and the constants are evaluated from the given conditions which is a valid approach for the initial-value case too.

Considering (8), the solution of the original problem is given by

$$u(x) = A + Bx + L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{9}$$

where A and B are integration constants that can be found from boundary or initial conditions. After substituting (2) in (9), the nonlinear term Nu is obtained as

$$Nu(x) = \sum_{i=0}^{\infty} A_i \tag{10}$$

where A_i are polynomials given by

$$A_i = \frac{1}{n!}\left[\frac{d^i}{d\lambda^i}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]\right]_{\lambda=0} \tag{11}$$

and referred to as Adomian polynomials [6]. After arranging the terms, the ADM obtains the solution as

$$\sum_{i=0}^{\infty} u_i(x) = u_0 - L^{-1}R\sum_{i=0}^{\infty} u_i(x) - L^{-1}\sum_{i=0}^{\infty} A_i \tag{12}$$

where

$$u_0 = A + Bx + L^{-1}g \tag{13}$$

and u_i are obtained as

$$\begin{aligned} u_1 &= -L^{-1}Ru_0 - L^{-1}A_0 \\ u_2 &= -L^{-1}Ru_1 - L^{-1}A_1 \\ &\vdots \\ u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n \end{aligned} \tag{14}$$

in a recursive form consequently (the interested reader is referred to [6], [11] for further details) . As the ADM suggests the above steps to obtain a solution for a general differential equation, in the case of partial differential operators, the above steps are valid for each one and a corresponding solution can be obtained. Such a solution is known as a partial solution, e.g. x-partial solution and y-partial solution in the case of a two dimensional problem involving independent variables x and y respectively [7].

Consider a boundary condition problem involving more than one independent variables. The partial solutions obtained for the separate equations for the highest-order linear operator terms are identical for the case in which the boundary conditions are general and asymptotically equal when the boundary conditions in one independent variable are independent of other variables. For the case, each equation is solved for an-n-term approximation, i.e. n-term partial sum, and then the partial solutions are combined yielding the solution of concern [7].

In cases where one operator annihilates the series in a finite number of terms, at least one partial solution may not satisfy the corresponding conditions. In order to proceed with the ADM, the initial condition should be expressed in an appropriate series expansion form without making a priori assumptions about the solution [8].

Since the method does not resort to linearization or assumption of weak nonlinearity, the solution generated is in general relatively realistic in the sense that the fidelity to the model of the physical problem is preserved.

III. Application to the One Group Neutron Diffusion Equations

We consider the time independent neutron diffusion equation for a homogenous region in a scenario where a geometry with the vacuum boundary conditions are valid:

$$\nabla^2\phi(\bar{\mathrm{r}}) - \kappa^2\phi(\bar{\mathrm{r}}) = -\frac{\mathrm{S}(\bar{\mathrm{r}})}{\mathrm{D}}, \quad \bar{\mathrm{r}} \in \mathrm{V} \tag{15}$$

$$\phi(\bar{\mathrm{r}}) = 0, \quad \bar{\mathrm{r}} \in \mathrm{S}$$

Here, $\phi(\vec{r})$ and $S(\vec{r})$ are the neutron flux and the neutron source term respectively. Σ_a is given in terms of the absorption cross section and the diffusion constant D by inverse diffusion length $\kappa^2 = \Sigma_a / D$. Note that, in one group criticality eigenvalue problems, the fission source term is $S = \nu\Sigma_f / Dk_{eff}$

We consider a two dimensional system with a square geometry. The system is symmetric with respect to both x and y axes and so it is sufficient to obtain a solution for only a single quadrant, for the case, using the ADM. In this scenario, the neutron diffusion equation together with the boundary conditions given by (14) reduces to

$$\frac{\partial^2\phi(x,y)}{\partial x^2}+\frac{\partial^2\phi(x,y)}{\partial x^2}-\frac{\Sigma_a}{D}\phi(x,y)=-\frac{\nu\Sigma_f}{Dk_{eff}}\phi(x,y) \quad (16)$$

$$\frac{\partial\phi(x,y)}{\partial x}=0 \quad at \quad x=0 \qquad \phi(x,y)=0 \quad at \quad x=a$$

$$\frac{\partial\phi(x,y)}{\partial y}=0 \quad at \quad y=0 \qquad \phi(x,y)=0 \quad at \quad y=a$$

After rewriting the above using an operator notation we obtain a similar form with that of (1) :

$$F\left[\phi(x,y)\right]=0 \quad (17)$$

$$F=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\chi^2 \quad (18)$$

$$L_x=\frac{\partial^2}{\partial x^2},\; R=\frac{\partial^2}{\partial y^2}+\chi^2,\; N=0 \quad (19)$$

where

$$\chi^2=\frac{\nu\Sigma_f}{Dk_{eff}}-\frac{\Sigma_r}{D} \quad (20)$$

Since (16) involves partial differential operators, ADM recursions are carried out for each of the independent variables yielding the x-partial solution and y-partial solution. Due to the boundary conditions [8], one subtlety here is that, due to these boundary conditions the x-partial solution of our equation will be same with y-partial solution. Hence, we consider the x partial solution and continue on for the recursion. After arrangements for collecting terms with the operator L_x on one side, we multiply the equality with inverse operator of L_x.

$$L_x^{-1}L_x\phi(x,y)=L_x^{-1}[\chi^2\phi(x,y)-L_y\phi(x,y)] \quad (21)$$

$$\phi(x,y)=A(y)x+B(y)+L_x^{-1}[\chi^2\phi(x,y)-L_y\phi(x,y)] \quad (22)$$

This yields the equation as seen where A(y) and B(y) terms are integral constants depending on the boundary conditions. The ADM recursion is as follows: ϕ_0 is composed of integral constants and the forcing term given by

$$\phi_0(x,y)=A(y)x+B(y) \quad (23)$$

If we apply first boundary conditions to ϕ_0 at x=0, we find out that *A(y)*=0 for all y. Then we continue with the ADM recursions as follows;

$$\begin{aligned}
&\phi_0(x,y)=B(y)\\
&\phi_1=L_x^{-1}[\chi^2\phi_0]-L_x^{-1}[L_y\phi_0]\\
&\phi_2=L_x^{-1}[\chi^2\phi_1]-L_x^{-1}[L_y\phi_1] \qquad (24)\\
&\vdots\\
&\phi_{n+1}=L_x^{-1}[\chi^2\phi_n]-L_x^{-1}[L_y\phi_n]
\end{aligned}$$

Now we will deal with the second term due to boundary conditions, i.e. B(y), and the forcing term. For this problem we have the case in which one operator annihilates in a finite number of steps as discussed in Section 2. Hence, straightforward application of ADM recursions lead both B(y) disappear even in the first step of the iteration that computes ϕ_1. This is the second subtlety. Here, in order to prevent loosing the contribution of these terms we represent both the integral constant B(y) and the forcing term with series expansions [8]. Note that this procedure requires no a priori assumptions on the solution. (Furthermore, in the case of a fission source, the series representation of the forcing term will be that of the source function.) For our fixed source case, we find out this form for beta m and f m through solving for the boundary conditions as in the following;

$$\mathrm{B(y)}=\sum_{n=0}^{\infty}b_n \mathrm{Cos}(\beta_n y)$$

Considering the type of the one group neutron diffusion equation and the boundary conditions, we consider an initial guess in the form. Then the first term of the series is given by

$$\phi_o(x,y)=\sum_{n=0}^{\infty}b_n \mathrm{Cos}(\beta_n y) \quad (25)$$

In order to have the presumption in (25) satisfy the boundary conditions, we substitute the condition for y = a yielding that β_n satisfies

$$\beta_n=\frac{(2n+1)\pi}{2a} \quad n=0,1,2,.... \quad (26)$$

On the other hand, consider a few terms of the series given by

$$\phi_o(x,y)=\sum_{n=0}^{\infty}b_n \mathrm{Cos}(\beta_n y) \quad (27)$$

$$\begin{aligned}
\phi_1(x,y)&=-\chi^2\iint_x \phi_0(x,y)dx'dx'-\iint_x\frac{\partial^2\phi_0(x,y)}{\partial y^2}dx'dx'\\
&=-\sum_{n=0}^{\infty}b_n\frac{(\alpha_n x)^2}{2!}Cos(\beta_n y)
\end{aligned}$$

$$\phi_2(x,y) = -\chi^2 \iint_x \phi_1(x,y)dx'dx' - \iint_x \frac{\partial^2 \phi_1(x,y)}{\partial y^2}dx'dx'$$

$$= \sum_{n=0}^{\infty} b_n \frac{(\alpha_n x)^4}{4!} Cos(\beta_n y)$$

$$\vdots$$

where

$$\alpha_n^2 = \chi^2 - \beta_n^2$$

Consider the partial sum of a few terms of the series given by

$$\phi(\mathrm{x,y}) = \sum_{\mathrm{m}=0}^{\infty} \phi_{\mathrm{m}}(\mathrm{x,y})$$

$$= \sum_{\mathrm{n}=0}^{\infty} \mathrm{b_n} \mathrm{Cos}\beta_{\mathrm{n}} \mathrm{y} \left(1 - \frac{(\alpha_{\mathrm{n}}\mathrm{x})^2}{2!} + \frac{(\alpha_{\mathrm{n}}\mathrm{x})^4}{4!} \cdots \right) \quad (28)$$

$$= \sum_{\mathrm{n}=0}^{\infty} \mathrm{b_n} \mathrm{Cos}(\beta_{\mathrm{n}}\mathrm{y}) \mathrm{Cos}(\alpha_{\mathrm{n}}\mathrm{x})$$

Applying (28) boundary condition at x=a, α_n is obtained

$$\alpha_n = \frac{(2n+1)\pi}{2a} \quad n = 0,1,2,.... \quad (29)$$

For a critical reactor, all the harmonics drop out and the fundamental eigenvalue is needed [19]-[21]. Under this condition, fundamental eigenvalue and eigenfunction given by

$$\alpha_0 = \frac{\pi}{2a\sqrt{\hbar}} \quad (30)$$

$$\phi(\mathrm{x,y}) = \mathrm{b_0} \mathrm{Cos}(\beta_0 \mathrm{y}) \mathrm{Cos}(\alpha_0 \mathrm{x})$$

and multiplication factor k_{eff} using (20), (26) and (30)

$$k_{eff} = \frac{\nu\Sigma_f}{D\left[\beta_0^2 - \alpha_0^2 + \frac{\Sigma_r}{D}\right]}$$

is obtained. But, coefficient b_0 is not determined yet.

In nuclear reactors, reactor power is determined following equation [21]

$$\mathrm{P} = \mathrm{w_f} \Sigma_{\mathrm{f}} \int_{-\mathrm{a}}^{\mathrm{a}} \int_{-\mathrm{a}}^{\mathrm{a}} \phi(\mathrm{x,y}) \mathrm{dx\,dy} \Rightarrow$$

$$\mathrm{b_0} = \frac{\mathrm{P}}{\mathrm{w_f} \Sigma_{\mathrm{f}} \int_{-\mathrm{a}}^{\mathrm{a}} \int_{-\mathrm{a}}^{\mathrm{a}} \mathrm{Cos}(\beta_0 \mathrm{y}) \mathrm{Cos}(\sqrt{\hbar}\alpha_0 \mathrm{x}) \mathrm{dx\,dy}} = \phi_0$$

IV. EXAMPLE

In this example, we consider a square reactor core with one edge length 2a = 100 cm and apply ADM for one quadrant of the system due to symmetricity. Notice the vacuum conditions at the left and upper boundary and reflector conditions at the right and lower boundary expressed in (16). The constants of reactor are presented in Table 1.

For the case, we assume the result obtained via the separation of variables as the exact solution. Computations utilizing Mathematica yield that ADM achieves this result as well. We present the computational results of ADM on a 100x100 grid in Figure 1 and on y=0 in Figure 2. In addition, in Table 2, we present the computational results of ADM, SoV in a comparative manner.

Table 1 – Reactor constants.

Constant	Value
a (cm)	50
D (cm)	1.77764
Σ_a (cm^{-1})	0.0143676
$\nu\Sigma_f$ (cm^{-1})	0.0262173
Σ_f (cm^{-1})	0.0104869
P (watt.cm^{-1})	32000
w_f (joule)	3.2042x10^{-11}

Table 2 – Compared results.

	ADM	SoV
k_{eff}	1.46657782	1.46657782
ϕ_0	2.34976x10^{13}	2.34976x10^{13}

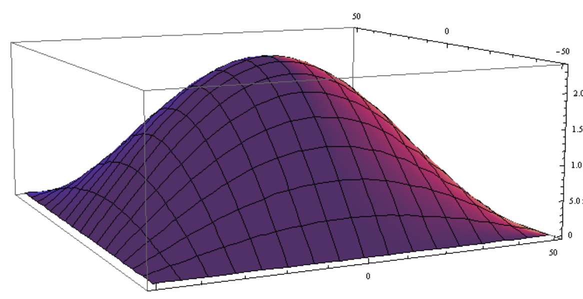

Figure 1 – Neutron flux distribution.

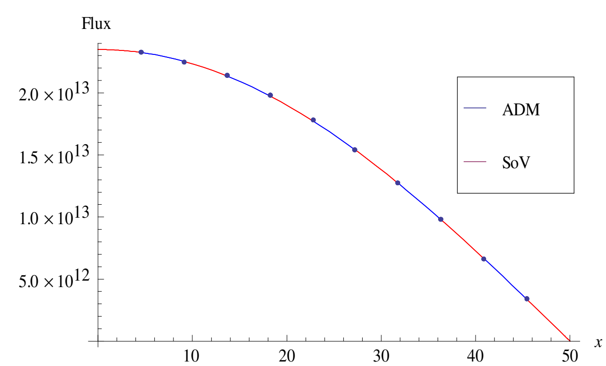

Figure 2 – Neutron flux for y=0.

V. CONCLUSION

We have considered two dimensional one group neutron diffusion equations for multiplying media and through the Adomian Decomposition Method we have obtained an iterative scheme for a series expansion of the solution. The iterations admit a symbolic program that outputs the solution as the partial sum of desired number of terms. It is also possible to perform numerical evaluation of the solution with a desired bit resolution. Also considering the fact that the closed form solution obtained through separation of variables is in a series expansion form, this approach provides an effective solution which also exhibits convenient numerical properties.

We calculate the eigenvalues, eigenfunctions and the largest eigenvalue named the multiplication factor keff . The computational results indicate that ADM coverges to the solution which the series sum provided by the widely used analytic method of SoV converges. This is in parallel with the behaviour of ADM for fixed source neutron diffusion equations. In this case, ADM yields a simple recursion with a competetive accuracy [18]. Similarly, we have obtained a straightforward solution for the case which provides motivation for exploiting ADM in a multiregion and/or multigroup scenario in which there are distinct diffusion constants inhibiting to achieve a solution through conventional approaches. A further investigation along these lines remains as future work.

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BIOGRAPHIES

Şükran Çavdar – received her B.S. degree in Electrical Engineering from Istanbul Technical University, İstanbul, Turkey. She was a teaching and research assistant at Institute of Nuclear Energy, ITU between 1996 and 2002. She received her Ph.D. degree in Nuclear Science from Institute of Energy, ITU. Her research interests include neutron diffusion calculations; analytic, semi-analytic and numerical solutions of linear, nonlinear differential and integro-differential equations.

She has publications on applications of boundary elements method, finite elements method, the BE/FE hybrid method and the Adomian decomposition method to one or multi-group neutron diffusion calculations. She has investigated the nonexistence of positive solutions to nonlinear parabolic differential equations. She is currently focused on chebyshev polynomial acceleration for the BE/FE hybrid method in multigroup neutron diffusion equations and ADM and HAM in multi-group neutron diffusion equations.