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Homotopy Analysis Method for One Group Neutron Diffusion Equation

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ABSTRACT – In this work, we solve the neutron diffusion equation for multiplying media using the Homotopy Analysis Method (HAM). HAM is proposed in 1992 by Shi Jun Liao and has been developed since then. It is based on a fundamental concept in differential geometry and topology, the homotopy. It has proved useful for problems involving algebraic, linear/non-linear, ordinary/partial differential and differential-integral equations being an analytic, recursive method that provides a series sum solution. It has the advantage of offering a certain freedom for the choice of its arguments such as the initial guess, the auxiliary linear operator and the convergence control parameter, and it allows us to effectively control the rate and region of convergence of the series solution. In this respect, this work is a part of our research motivated by the question of whether methods for solving the neutron diffusion equation that yield straightforward expressions but able to provide a solution of reasonable accuracy exist such that we could avoid analytic methods that are widely used but either fail to solve the problem or provide solutions through many intricate expressions that are likely to contain mistakes or numerical methods that require powerful computational resources and advanced programming skills due to their very nature or intricate mathematical fundamentals. Regarding the utilization of HAM, we employ Fourier basis for expressing the initial guess due to the structure of the problem and its boundary conditions. We present the results in comparison with other widely used method of Separation of Variable (SoV).

Keywords: *homotopy analysis method, neutron diffusion equation*

I. INTRODUCTION

The Homotopy Analysis Method (HAM) is an analytic method that provides series solutions and has been proposed first by Liao [1]. It has been successfully applied to linear and non-linear equations in various fields of engineering and science since then (e.g. [2-12]). The features of this approach includes that the convergence speed and the region of convergence (ROC) of the series solution can be controlled via convergence control parameters h, which is an advantage. In this respect, it is possible to utilize HAM for linear and/or non-linear problems without any assumptions or restrictions [6].

Another appeal of HAM is that, there is a broad range of freedom for the selection of the base functions suitable for the problem in hand, the initial guess, the auxiliary linear operator and the convergence control parameters. However, this selection is by no means arbitrary and depends on the type of the problem as well as its boundary conditions. A thorough treatment of this issue can be found in [6].

In a recent work, we have solved the fixed source neutron diffusion equation using HAM [13]. Here, we consider a multiplying media scenario and apply the HAM to one group neutron diffusion equation. In Section 2 we give the definitions that are fundamental to this method. We next consider neutron diffusion equations and utilize HAM for solving. The results are presented in Section 4 in comparison to the Seperation of Variable (SoV) method. Finally, we conclude.

II. FUNDAMENTAL DEFINITION

The Homotopy Analysis Method (HAM) considers a general set of equations involving linear and/or non-linear operators N_i such that

$$N_i\left[u_i(\vec{x})\right] = g_i(\vec{x}), \quad i = 1,..,l \tag{1}$$

for i = 1, 2, ..., l where $\vec{x}$ is an independent variable, $g_i(\vec{x})$ is a known function and $\{u_i(\vec{x})\}_{i=1,2,...l}$ is the solution[6]. Let us consider $u_i(\vec{x})$; its so-called zero-order deformation equation is constructed, through an unknown function $\Theta_i(\vec{x};q)$ as

$$(1-q)\pounds\left[\Theta_i(\vec{x};q) - u_{i,0}(\vec{x})\right] = q\hbar\left\{N\left[\Theta_i(\vec{x};q)\right] - g_i(\vec{x})\right\} \tag{2}$$

where q ∈ [0, 1] is the embedding parameter, $\hbar$ is a non-zero real number referred to as the auxiliary parameter or the convergence control parameter, £ is the auxiliary linear operator, and $u_{i,0}(\vec{x})$ is an initial guess for the solution $u_i(\vec{x})$.

For q = 0, the zero-order deformation equation given by (2) leads

$$\Theta_i(\vec{x};0) = u_{i,0}(\vec{x}) \tag{3}$$

and it is apparent that the initial guess, i.e. $u_{i,0}(\vec{x})$, does not rely on the boundary conditions of the problem. It is important, that one has a remarkable freedom in choosing the initial guess and other auxiliary parameters.

On the other hand, for q = 1, (2) leads

$$\Theta_i(\vec{x};1) = u_i(\vec{x}) \tag{4}$$

which together with (3) reveals that as q varies from 0 to 1, or as the equation of consideration together with the corresponding solution are deformed -in a sense-, the solution of (2) varies from $u_{i,0}(\vec{x})$ to $u_i(\vec{x})$. Hence, in order to proceed with HAM, it is necessary to find $\Theta_i(\vec{x};q)$.

The deformation of $\Theta_i(\vec{x};q)$ corresponding to q increasing from 0 to 1 can be obtained entirely through (2). For this, first consider the Taylor series expansion of $\Theta_i(\vec{x};q)$ with respect to q which in turn leads the Homotopy series given by

$$\Theta_i(\vec{x};q) = u_{i,0}(\vec{x}) + \sum_{m=1}^{\infty} u_{m,i}(\vec{x}) q^m \tag{5}$$

where

$$u_{m,i}(\vec{x}) = \frac{1}{m!} \frac{\partial^m \Theta_i(\vec{x};q)}{\partial q^m}\bigg|_{q=0} \tag{6}$$

Moreover, appropriate selections of $u_{i,0}(\vec{x})$, $\hbar$ and £, yields a convergent series for q = 1 [6]-[11], i.e.

$$\Theta_i(\vec{x};1) = u_{i,0}(\vec{x}) + \sum_{m=1}^{\infty} u_{i,m}(\vec{x}) = u_i(\vec{x}) \tag{7}$$

The series above is one of the solutions of the original set of equations as proved in [6] which also contains a thorough discussion of the method. Hence, in order to obtain a series solution for (1), it is sufficient to determine the $u_{i,m}(\vec{x})$ terms in (7).

In order to do this, HAM considers the mth order deformation equation, and the fundamental recursion of the method is obtained by m consecutive differentiation of (2) as follows:

$$£\left[u_{i,m}(\vec{x}) - \chi_m u_{i,m-1}(\vec{x})\right] = \hbar R_i(\vec{u}_{i,m-1}) \tag{8}$$

where R_i and χ_m are given by

$$R_i(\vec{u}_{i,m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\left\{N_i\left[\Theta_i(\vec{x};q)\right] - g_i(\vec{x})\right\}}{\partial q^{m-1}}\bigg|_{q=0} \tag{9}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} \tag{10}$$

respectively.

Utilizing this recursion, it is possible to find $u_{i,m}(\vec{x})$ s and hence a solution to the original set of equations through the recursion implied by (8).

III. Application to the One Group Neutron Diffusion Equations

The neutron diffusion equation for a homogenous reactor regarding a geometry where the vacuum boundary conditions are valid is given by

$$\nabla^2\phi(\vec{r}) - \kappa^2\phi(\vec{r}) = -\frac{S(\vec{r})}{D}, \quad \vec{r} \in V \tag{11}$$

$$\phi(\vec{r}) = 0, \quad \vec{r} \in S$$

where $\phi(\vec{r})$ is the neutron flux and $S(\vec{r})$ is the neutron source. Σ_a is given in terms of the absorption cross section and the diffusion constant D by inverse diffusion length $\kappa^2 = \Sigma_a / D$. In one group criticality eigenvalue problems, the source term named fission source is $S = \nu\Sigma_f / Dk_{eff}$

We consider the one group neutron diffusion equation for a two dimensional system with a square geometry. Since the system is symmetric with respect to both the x and y axes, we utilize HAM for only a single quadrant. For the case, the neutron diffusion equation together with the boundary conditions given by (11) reduces to

$$\frac{\partial^2\phi(x,y)}{\partial x^2} + \frac{\partial^2\phi(x,y)}{\partial x^2} - \frac{\Sigma_a}{D}\phi(x,y) = -\frac{\nu\Sigma_f}{Dk_{eff}}\phi(x,y) \tag{12}$$

$$\frac{\partial\phi(x,y)}{\partial x} = 0 \quad at \quad x = 0 \qquad \phi(x,y) = 0 \quad at \quad x = a$$

$$\frac{\partial\phi(x,y)}{\partial y} = 0 \quad at \quad y = 0 \qquad \phi(x,y) = 0 \quad at \quad y = a$$

It is possible to rewrite the above through an operator notation in similar form with that of (1), and achieve

$$N\left[\phi(x,y)\right] = 0 \tag{13}$$

$$N = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} + \chi^2 \tag{14}$$

where

$$\chi^2 = \frac{\nu\Sigma_f}{Dk_{eff}} - \frac{\Sigma_r}{D} \tag{15}$$

Hence, the zero-order deformation equation for the one group neutron equation is obtained as

$$(1-q)£\left[\Theta(x,y;q) - \phi_0(x,y)\right] = q\hbar\left\{N\left[\Theta(x,y;q)\right]\right\} \tag{16}$$

It is apparent that, the initial guess $\phi_0(x,y)$ and the auxiliary linear operator £ should be determined first in (16).

Considering the type of the one group neutron diffusion equation and the boundary conditions, we consider an initial guess in the form

$$\phi_o(x,y)=\sum_{n=0}^{\infty} b_n \mathrm{Cos}(\beta_n y) \tag{17}$$

In order to have the presumption in (16) satisfy the boundary conditions, we substitute the condition for y = a yielding that β_n satisfies

$$\beta_n=\frac{(2n+1)\pi}{2a} \quad n=0,1,2,.... \tag{18}$$

Regarding the selection of the auxiliary linear operator £, it is apparent that N given by (16) is a good candidate and hence

$$£=\frac{\partial^2}{\partial x^2} \tag{19}$$

We obtain the HAM recursion through m consecutive differentiation of the corresponding zero-order deformation equation, i.e. (16), as

$$£\left[\phi_m(x,y)-\chi_m\phi_{m-1}(x,y)\right]=\hbar R_m(\vec{\phi}_{m-1}) \tag{20}$$

and consequently

$$\phi_m(x,y)=\chi_m\phi_{m-1}(x,y)+£^{-1}\left[\hbar R_m(\vec{\phi}_{m-1})\right] \tag{21}$$

for m > 0. The R_m term above is obtained through (9) and (14) as

$$R_m=\frac{\partial^2\phi_{m-1}(x,y)}{\partial x^2}+\frac{\partial^2\phi_{m-1}(x,y)}{\partial y^2}+\chi^2\phi_{m-1}(x,y)$$

It is possible to exploit (21) in order to utilize the recursion given by (20) through software packages that feature symbolic programming such as Mathematica and Matlab.

On the other hand, consider a few terms of the series given by

$$\phi_o(x,y)=\sum_{n=0}^{\infty} b_n \mathrm{Cos}(\beta_n y) \tag{22}$$

$$R_1=\frac{\partial^2\phi_o(x,y)}{\partial x^2}+\frac{\partial^2\phi_o(x,y)}{\partial y^2}+\chi^2\phi_o(x,y)$$

$$\phi_1(x,y)=\hbar\iint_x R_1(\vec{\phi}_o)dx'dx'$$

$$=-\sum_{n=0}^{\infty} b_n\hbar\frac{(\alpha_n x)^2}{2!}Cos(\beta_n y)$$

where

$$\alpha_n^2=\beta_n^2-\chi^2$$

$$R_2=\frac{\partial^2\phi_1(x,y)}{\partial x^2}+\frac{\partial^2\phi_1(x,y)}{\partial y^2}+\chi^2\phi_1(x,y)$$

$$\phi_2(x,y)=\phi_1(x,y)+\hbar\iint_x R_2(\vec{\phi}_1)dx'dx'$$

$$=\sum_{n=0}^{\infty} b_n\hbar^2\frac{(\alpha_n x)^4}{4!}Cos(\beta_n y)$$

$$\vdots$$

Consider the partial sum of a few terms of the series given by

$$\phi(x,y)=\sum_{m=0}^{\infty}\phi_m(x,y)$$

$$=\sum_{n=0}^{\infty} b_n \mathrm{Cos}\beta_n y\left(1-\hbar\frac{(\alpha_n x)^2}{2!}+\hbar^2\frac{(\alpha_n x)^4}{4!}\cdots\right) \tag{23}$$

$$=\sum_{n=0}^{\infty} b_n \mathrm{Cos}(\beta_n y)\mathrm{Cos}(\sqrt{\hbar}\alpha_n x)$$

Applying (23) boundary condition at x=a, α_n is obtained

$$\alpha_n=\frac{(2n+1)\pi}{2a\sqrt{\hbar}} \quad n=0,1,2,.... \tag{24}$$

For a critical reactor, all the harmonics drop out and it is sufficient to consider the fundamental eigenvalue [14]-[16]. Therefore, the fundamental eigenvalue and eigenfunction given by

$$\alpha_0=\frac{\pi}{2a\sqrt{\hbar}} \tag{25}$$

$$\phi(x,y)=b_0\mathrm{Cos}(\beta_0 y)\mathrm{Cos}(\sqrt{\hbar}\alpha_0 x)$$

and multiplication factor k_{eff} using (15), (18) and (25)

$$k_{eff}=\frac{\nu\Sigma_f}{D\left[\beta_0^2-\alpha_0^2+\frac{\Sigma_r}{D}\right]} \tag{26}$$

is obtained.

Notice that $\hbar$ enables the control of the convergence of the series sum. It cancels out in (25) and the series sum converges for all values of $\hbar$ for the problem of concern.

In order to determine the coefficient b_0 we consider the fact that in nuclear reactors, reactor power is determined by the following equation [16]

$$P=w_f\Sigma_f\int_{-a}^{a}\int_{-a}^{a}\phi(x,y)dx\,dy \Rightarrow$$

$$b_0=\frac{P}{w_f\Sigma_f\int_{-a}^{a}\int_{-a}^{a}\mathrm{Cos}(\beta_0 y)\mathrm{Cos}(\sqrt{\hbar}\alpha_0 x)dx\,dy}=\phi_0 \tag{27}$$

IV. EXAMPLE

In this example, we consider a square reactor core with edge length 2a = 100 cm and apply HAM for one quadrant of the system which is sufficient owing to the symmetricity. Notice that the vacuum conditions at the left and upper boundaries together with the reflector conditions at the right and lower boundaries are as expressed in (12). The constants of the reactor are presented in Table 1.

Table 1 – Reactor constants.

Constant	Value
a (cm)	50
D (cm)	1.77764
Σ_a (cm^{-1})	0.0143676
$\nu\Sigma_f$ (cm^{-1})	0.0262173
Σ_f (cm^{-1})	0.0104869
P (watt.cm^{-1})	32000
w_f (joule)	3.2042x10^{-11}

For the case, we consider the result obtained via separation of variables as the exact solution. Computations employing Mathematica yield that HAM converges to the exact solution. We present the computational results of HAM on a 100x100 grid in Figure 1 and on y=0 in Figure 2. In addition, in Table 2, we present the computational results of HAM and SoV in a comparative manner.

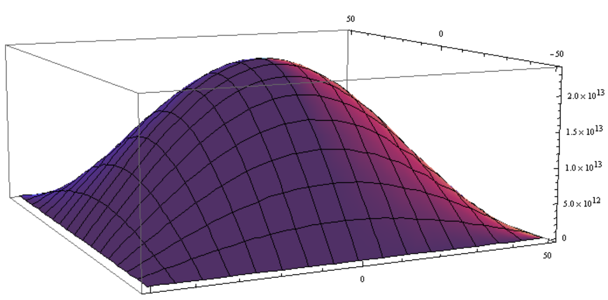

Figure 1 – Neutron flux distribution.

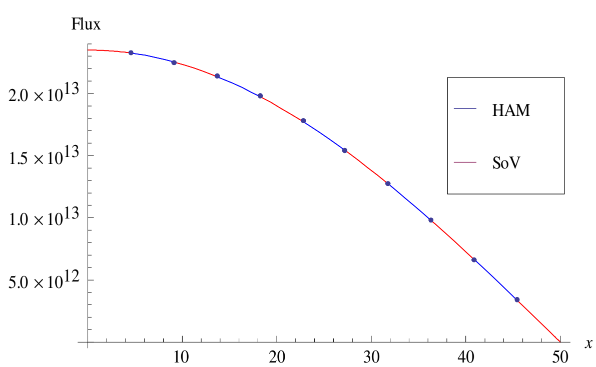

Figure 2 – Neutron flux for y=0.

Table 2 – Compared results.

	HAM	SoV
k_{eff}	1.46657782	1.46657782
ϕ_0	2.34976x10^{13}	2.34976x10^{13}

V. CONCLUSION

In this work, we solve the one group neutron diffusion equations for multpliying media using the Homotopy Analysis Method. We calculate the eigenvalues, eigenfunctions and the largest eigenvalue named multiplication factor k_{eff} . The computational results indicate that the iterative approach of HAM converges to the true result provided by the widely used analytic method of SoV. This also holds when HAM is applied for the fixed source neutron diffusion equations for which case the HAM produces the result in a rather straightforward manner compared to that of the SoV approach which is yield through tedious algebraic manipulations of complicated mathematical expressions [13]. Similarly, for the problem of concern, we have obtained a simple iteration using HAM. We believe that these results are promising in that they provide a strong motivation for exploiting HAM in multigroup and/or multiregion neutron diffusion equations: In both cases, the existence of distinct regions and/or energy groups yield differing diffusion constants which in turn render a solution through convention approaches not possible, in general. It is our expectation that HAM would provide advantages in such scenarios and its further elaboration remains as future work.

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