Solution of Fixed Source Neutron Diffusion Equation via Homotopy Perturbation Method

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Abstract- In this work, Homotopy Perturbation Method (HPM), which is a combination of a homotopy transformation and the perturbation method is applied to the fixed source neutron diffusion equation. Regarding the utilization of HPM, we employ Fourier basis for expressing the initial approximation due to the structure of the problem and its boundary conditions. We compare the results of the present method with that obtained by the well known Separation of Variables (SoV).

Index Terms- Homotopy Perturbation Method, Fixed Source Neutron Diffusion Equation.

I. Introduction

HOMOTOPY Perturbation Method (HPM), proposed by J.H.He, does not require a small parameter like traditional perturbation methods [1]-[3]. The HPM has been studied extensively in both analytical and numerical aspects for solving linear and nonlinear differential and integral equations. It is a coupling of the homotopy and the perturbation technique. It continuously deforms the original equation to a simpler problem that yields a relatively straightforward solution.

In recent studies, iterations converging to the solution of the fixed source neutron diffusion equation (NDE) has been obtained using HAM and ADM [4], [5]. Here, we apply the HPM to NDE; in Section 2 we give the definitions that are fundamental to this method. We next consider NDE and utilize HPM. The results are presented in Section 4 in comparison to the Seperation of Variables (SoV) before we conclude.

II. Homotopy perturbation method

We summarize HPM as presented in [2] for convenience. In HPM, we consider an equation involving linear and/or nonlinear operators A, such that

$$A\left[u(\vec{r})\right]-f(\vec{r})=0, \quad \vec{r}\in\Omega \tag{1}$$

with boundary condition

$$B(u,\partial u/\partial n)=0, \quad \vec{r}\in\Gamma \tag{2}$$

where $\vec{r}$ is an independent variable, A is a general differential operator, B is a boundary operator, $f(\vec{r})$ is a known function and $u(\vec{r})$ is the solution. The operator A can be divided into linear (L) and nonlinear (N) parts.

Let us consider a homotopy equation which is constructed, through an unknown function $v(\vec{r};p)$ as

$$(1-p)\left\{\mathrm{L}\left[v(\vec{r};p)\right]-L\left[u_0(\vec{r})\right]\right\}+p\left\{A\left[v(\vec{r};p)\right]-f(\vec{x})\right\}=0 \tag{3}$$

where p∈ [0, 1] is an embedding parameter, $u_0(\vec{r})$ is an initial approximation of (1). For p = 0, the homotopy equation given by (3) leads

$$v(\vec{r};0)-u_0(\vec{r})=0 \tag{4}$$

and we note that the initial approximation is a distinct concept than the boundary conditions of the problem.

On the other hand, for p = 1, (3) leads

$$A\left[v(\vec{r};1)\right]-f(\vec{r})=0 \tag{5}$$

As p is varied from zero to unity, $v(\vec{r};p)$ varies from $u_0(\vec{r})$ to $u(\vec{r})$. In topology, this is called deformation, and (4) and (5) are called homotopic.

Assume that the solution of (3) can be written as a power series in p

$$v(\vec{r};p)=v_0+pv_1+p^2v_2+p^3v_3+.... \tag{6}$$

Setting p=1 results in the approximate solution of (1)

$$u=\lim_{p\to1}v=v_0+v_1+v_2+v_3+.... \tag{7}$$

The convergence of the series above has been proved in [2].

III. APPLICATION TO THE FIXED SOURCE NDE

The neutron diffusion equation (NDE) for a general geometry where the vacuum boundary conditions are valid is given by

$$\nabla^2\phi(\vec{\mathrm{r}})-\kappa^2\phi(\vec{\mathrm{r}})=-\frac{\mathrm{S}(\vec{\mathrm{r}})}{\mathrm{D}}, \quad \vec{\mathrm{r}}\in\mathrm{V}; \quad \phi(\vec{\mathrm{r}})=0, \ \vec{\mathrm{r}}\in\mathrm{S} \tag{8}$$

where ϕ is the neutron flux, S is the neutron source, D is the diffusion constant, Σ_a is the absorption cross section and $\kappa^2=\Sigma_\mathrm{a}/\mathrm{D}$.

We consider the NDE for a two dimensional system with a square geometry. Since the system is symmetric with respect to both the x and y axes, we utilize HPM for only a single quadrant. For the case, the NDE together with the boundary conditions given by (8) reduces to

$$\frac{\partial^2\phi(x,y)}{\partial x^2}+\frac{\partial^2\phi(x,y)}{\partial x^2}-\kappa^2\phi(x,y)=-\frac{S}{D} \tag{9}$$

$$\frac{\partial\phi(x,y)}{\partial x}=0 \quad at \quad x=0 \qquad \phi(x,y)=0 \quad at \quad x=a$$

$$\frac{\partial\phi(x,y)}{\partial y}=0 \quad at \quad y=0 \qquad \phi(x,y)=0 \quad at \quad y=a$$

In order to utilize the HPM, we substitute $L = \partial^2/\partial x^2$ and $A = \partial^2/\partial x^2 + \partial^2/\partial y^2 - \kappa^2$ in (9) and construct the homotopy equation which has the general form given by (3), i.e.

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 \phi_0}{\partial x^2} + p\frac{\partial^2 \phi_0}{\partial x^2} + p\frac{\partial^2 v}{\partial y^2} - p\kappa^2 v + p\frac{S}{D} = 0 \qquad (10)$$

Regarding the type of the fixed source neutron diffusion equation and the boundary conditions, we consider an initial approximation in the form

$$\phi_o(x, y) = \sum_{n=0}^{\infty} b_n \mathrm{Cos}(\beta_n y) \qquad (11)$$

In order to have the presumption in (11) satisfy the boundary conditions, we substitute the condition for y = a yielding that $\beta_n = (2n+1)\pi/2a \quad n = 0,1,2,....$

For convenience, we rewrite the known force term in (10) in the same basis set with (11), i.e.

$$\frac{S}{D} = \sum_{n=0}^{\infty} s_n \mathrm{Cos}(\beta_n y) \qquad (12)$$

The orthogonality of the Fourier basis yields that $s_n = (-1)^n 2S/aD\beta_n$ holds.

We obtain the HPM recursion after substituting the presumption given by (6) in (10):

$$\begin{aligned}
&p^0 : \frac{\partial^2 v_o(x,y)}{\partial x^2} = 0 \\
&p^1 : \frac{\partial^2 v_1(x,y)}{\partial x^2} + \frac{\partial^2 v_o(x,y)}{\partial y^2} - \kappa^2 v_o(x,y) + \frac{S}{D} = 0 \\
&p^2 : \frac{\partial^2 v_2(x,y)}{\partial x^2} + \frac{\partial^2 v_1(x,y)}{\partial y^2} - \kappa^2 v_1(x,y) = 0 \\
&\vdots
\end{aligned} \qquad (13)$$

The coupled equations above are solved starting from v_0 and proceeding in order. The corresponding solutions of the first two equations are given as

$$\begin{aligned}
&v_1(x,y;p) = p^0 \sum_{n=0}^{\infty} b_n Cos(\beta_n y)\left[\alpha_n^2 \frac{x^2}{2!} - \frac{2S}{Da}\frac{(-1)^n}{\beta_n}\frac{x^2}{2!}\right] \\
&v_2(x,y;p) = p^2 \sum_{n=0}^{\infty} b_n Cos(\beta_n y)\left[\alpha_n^4 \frac{x^4}{4!} - \alpha_n^2 \frac{2S}{Da}\frac{(-1)^n}{\beta_n}\frac{x^4}{4!}\right] \\
&\vdots
\end{aligned} \qquad (14)$$

where $\alpha_n^2 = \beta_n^2 - \kappa^2$. Considering (6) and (7) solution of the NDE is given by

$$\begin{aligned}
\phi(x,y) &= \sum_{m=0}^{\infty} \phi_m(x,y) \\
&= \frac{2S}{Da}\sum_{n=0}^{\infty} \frac{(-1)^n}{\beta_n \alpha_n^2} \mathrm{Cos}\beta_n y \left(1 - \frac{\mathrm{Cosh}(\alpha_n x)}{\mathrm{Cosh}(\alpha_n a)}\right)
\end{aligned} \qquad (15)$$

The recursions above can be realized through software packages that feature symbolic programming such as Mathematica and Matlab.

IV. Example

In this example, we consider a square reactor core with edge length 2a = 50 cm and apply HPM for one quadrant of the system which is sufficient owing to the symmetricity. Notice that the vacuum conditions at the left and upper boundaries together with the reflector conditions at the right and lower boundaries are as expressed in (9). The constants of the reactor are presented in Table 1.

TABLE I
REACTOR CONSTANTS

Constant	Value
a (cm)	25
D (cm)	1.77764
Σ_a (cm^{-1})	1
S	1

For the case, the series sum obtained via the separation of variables achieves 10^{-7} precision for the partial sum of the first M = 356 terms. We assume this result as the exact solution. Computations utilizing Mathematica yield that HPM achieves this precision for M = 356 terms partial sum as well.

We present the computational results of HPM on a 25x25 grid and for M = 356 term partial sum in Table 2 together with that for SoV in a comparative manner.

TABLE II
RESULTS FOR Y=0

x (cm)	SoV	HPM	Error ($x10^{-9}$)
0	47.47293	47.47293	5
5	46.32885	46.32885	5
10	42.57155	42.57155	6
15	35.15265	35.15265	7
20	22.07556	22.07556	1.1
25	0	0	0

V. Conclusion

In this work, we apply the HPM to fixed source neutron diffusion equation. The computational results indicate that HPM, compared to the widely used analytic method of separation of variables, yields shorter and relatively straightforward expressions for the solution and exhibits high accuracy with a comparable convergence speed.

VI. References

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