Multigroup Finite Element-Boundary Element Method For Neutron Diffusion

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- Why is the Hybrid Method preferable for Neutron Diffusion Computations?
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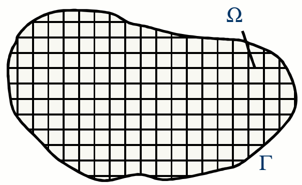
Introduction to the Hybrid Method:

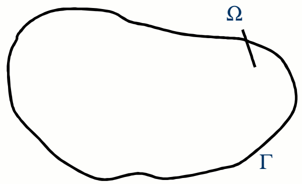
Advantages of FEM



- Applicable in the case of both regular and irregular geometries.
- Coefficients Matrix is symmetric and sparse.
- High rates of convergence is attainable by using high order elements.

Introduction to the Hybrid Method:

Advantages of BEM



- The dimensions of the geometry is decreased by one.
- The number of unknowns is decreased.
- Applicable for infinite and semi-infinite mediums.
- Applicable to irregular geometries.
- Easily adapted to changes of the geometry.

Introduction to the Hybrid Method:

Disadvantages of BEM/FEM

- Disadvantages of BEM
  - Hard to apply to inhomogeneous medium cases.
  - Non-sparse, non-symmetric Coefficients Matrix.
  - Subtleties of the mathematical background.
- Disadvantages of FEM
  - Poor scalability: Fast increase of number of unknowns and coefficient matrix dimensions in the number of elements.
  - Elements need to be redefined in case of changes in the geometry.

Why a Hybrid Method ?

is preferable for the Neutron Diffusion Computations.

- Neutron Diffusion Equations include a fission source.
- Internal flux distribution values need to be known since ND Equations define an eigenvalue-eigenvector problem

Discretization with HM:

Equations of the Core and the Reflector

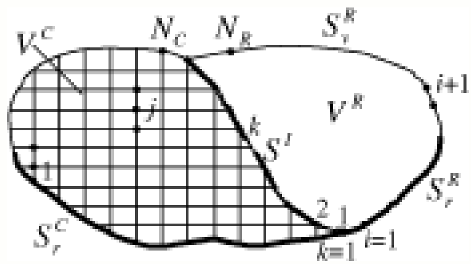
The within group diffusion equation of group *g* can be written as C(Core) and R(Reflector);

$$\nabla^2 \Phi_g^j(\vec{r}) - K_{j,g}^2 \Phi_g^j(\vec{r}) = -\frac{S_g^j(\vec{r})}{D_g^j} \qquad \vec{r} \in V^j,\ j = c \text{ or } r$$

Subject to the reflective or vacuum boundary conditions given below

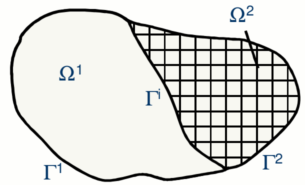
$$\vec{J}_g^-(\vec{r}) = \frac{\Phi_g(\vec{r})}{4} + \frac{D_g}{2}\frac{\partial \Phi_g(\vec{r})}{\partial n} = 0 \quad \vec{r} \in \Gamma_v^r \cup \Gamma_v^c \qquad \vec{J}_g^c(\vec{r}) = 0 \quad \vec{r} \in \Gamma_r^r \cup \Gamma_r^c$$

$$\Phi_g^c(\vec{r}) = \Phi_g^r(\vec{r}), \qquad \vec{J}_g^r(\vec{r}) = -\vec{J}_g^c(\vec{r}) \qquad \vec{r} \in \Gamma_i$$



Discretization with HM:

FEM Formulation for the Core



$$\int_{V^c}\left[\vec{\nabla}w(\vec{r})\cdot\vec{\nabla}\Phi_g(\vec{r})+K_{c,g}^2 w(\vec{r})\Phi_g(\vec{r})-w(\vec{r})\frac{S_g(\vec{r})}{D_g^c}\right]dV+\frac{1}{2D_g^c}\int_{S_v^c} w(\vec{r})\Phi_g(\vec{r})dS$$

$$+\frac{1}{D_g^c}\int_{S^I} w(\vec{r})J_g(\vec{r})dS=0$$

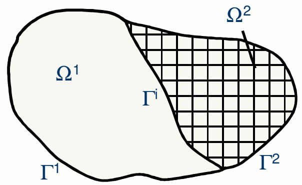
$$w(\vec{r})=(\underline{w}^c)^T\underline{h}^c(\vec{r})+(\underline{w}^I)^T\underline{h}^I(\vec{r})\qquad \Phi_g(\vec{r})=(\underline{\Phi}_g^c)^T\underline{h}^c(\vec{r})+(\underline{\Phi}_g^I)^T\underline{h}^I(\vec{r})\qquad J_g(\vec{r})=(\underline{J}_g^I)^T\underline{h}^I(\vec{r})$$

$\Rightarrow$

$$\underline{\underline{A}}_g^C\underline{\phi}_g^C+\underline{\underline{A}}_g^{CI}\underline{\phi}_g^I=\underline{S}_g^C \quad\longrightarrow\quad \left(\underline{\underline{A}}_g^{CI}\right)^T\underline{\phi}_g^C+\underline{\underline{A}}_g^I\underline{\phi}_g^I+\underline{\underline{A}}_g^{IJ}\underline{J}_g^I=\underline{S}_g^I$$

Discretization with HM:

Source Terms



Source terms;

$$\underline{q}_g^{c} = \sum_{g'=1}^{g-1}\left[\underline{\underline{S}}^{c}_{g\leftarrow g'}\underline{\Phi}^{c}_{g'} + \underline{\underline{S}}^{ci}_{g\leftarrow g'}\underline{\Phi}^{i}_{g'}\right] + \frac{1}{k_{eff}}\sum_{g'=1}^{G}\left[\underline{\underline{F}}^{c}_{g\leftarrow g'}\underline{\Phi}^{c}_{g'} + \underline{\underline{F}}^{ci}_{g\leftarrow g'}\underline{\Phi}^{i}_{g'}\right]$$

$$\underline{q}_g^{i} = \sum_{g'=1}^{g-1}\left[\underline{\underline{S}}^{i}_{g\leftarrow g'}\underline{\Phi}^{i}_{g'} + \left(\underline{\underline{S}}^{ci}_{g\leftarrow g'}\right)\underline{\Phi}^{c}_{g'}\right] + \frac{1}{k_{eff}}\sum_{g'=1}^{G}\left[\underline{\underline{F}}^{i}_{g\leftarrow g'}\underline{\Phi}^{i}_{g'} + \left(\underline{\underline{F}}^{ci}_{g\leftarrow g'}\right)^{T}\underline{\Phi}^{c}_{g'}\right]$$

Discretization with HM:

Boundary Integral Equation for Reflector

$$\int_{V^r}\left(\nabla^2\Phi_g^r(\vec{r})-\chi_{r,g}^2\Phi_g^r(\vec{r})+\frac{1}{D_g}\int_{V^r}S_g^r(\vec{r})\right)G_g(\vec{r},\vec{\rho})dV=0$$

Apply Green's Theorem twice

$$\int_{V^r}\left(\nabla^2 G_g(\vec{r},\vec{\rho})-\chi_{r,g}^2 G_g(\vec{r},\vec{\rho})\right)\Phi_g^r(\vec{r})dV$$

$$=-\frac{1}{D_g}\int_{V^r}S_g^r(\vec{r})G_g(\vec{r},\vec{\rho})dV+\int_{\Gamma_{v+r+i}^r}\frac{\partial G_g(\vec{r},\vec{\rho})}{\partial n}\Phi_g^r(\vec{r})dV$$

$$-\int_{\Gamma_{v+r+i}^r}\frac{\partial\Phi_g^r(\vec{r})}{\partial n}G_g(\vec{r},\vec{\rho})dV$$

Discretization with HM:

Green's Function for the Infinite Medium

$$\nabla^2 G_g(\vec{r},\vec{\rho}) - \chi^2_{r,g} G_g(\vec{r},\vec{\rho}) = -\delta(\vec{r}-\vec{\rho})$$

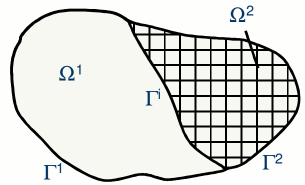
$$\int_V \delta(\vec{r}-\vec{\rho})\Phi_g(\vec{r})dV = \begin{cases} \Phi_g(\vec{r}) & \vec{\rho} \in V \wedge \vec{\rho} \notin S \\ \dfrac{\Theta(\vec{\rho})}{2\pi}\Phi_g(\vec{r}) & \vec{\rho} \in S \end{cases}$$

$$\frac{\Theta(\vec{\rho})}{2\pi}\Phi^r_g(\vec{r})dV + \int_{\Gamma^r_{v+r+i}} \frac{\partial \Phi^r_g(\vec{r})}{\partial n} G_g(\vec{r},\vec{\rho})dV - \int_{\Gamma^r_{v+r+i}} \frac{\partial G_g(\vec{r},\vec{\rho})}{\partial n} \Phi^r_g(\vec{r})dV$$

$$= -\frac{1}{D_g}\int_{V^r} S^r_g(\vec{r}) G_g(\vec{r},\vec{\rho})dV$$

Discretization with HM:

BEM Formulation



$$c(\vec{\rho})\phi_g(\vec{\rho})+\int_{S_r^R}\phi_g(\vec{r})\frac{\partial G_g}{\partial n}(\vec{r},\vec{\rho})dS+\int_{S_v^R}\phi_g(\vec{r})\left[\frac{G_g(\vec{r},\vec{\rho})}{2D_g^R}+\frac{\partial G_g}{\partial n}(\vec{r},\vec{\rho})\right]dS$$

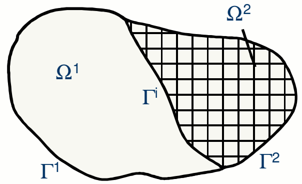
$$+\int_{S^I}\left[\phi_g(\vec{r})\frac{\partial G_g}{\partial n}(\vec{r},\vec{\rho})+\frac{G_g(\vec{r},\vec{\rho})}{2D_g^R}J_g(\vec{r})\right]dS=s_g(\vec{\rho}),\ \ \vec{\rho}\in V^R$$

$$s_g(\vec{\rho})=\sum_{g'=1}^{g-1}s_{g,g'}\left\{c(\vec{\rho})\phi_{g'}(\vec{\rho})+\int_{S_r^R}\phi_{g'}(\vec{r})\frac{\partial G_g}{\partial n}(\vec{r},\vec{\rho})dS+\int_{S_v^R}\phi_{g'}(\vec{r})\left[\frac{G_g(\vec{r},\vec{\rho})}{2D_g^R}+\frac{\partial G_g}{\partial n}(\vec{r},\vec{\rho})\right]dS\right.$$

$$\left.+\int_{S^I}\left[\phi_{g'}(\vec{r})\frac{\partial G_g}{\partial n}(\vec{r},\vec{\rho})+\frac{G_g(\vec{r},\vec{\rho})}{D_g^R}J_{g'}(\vec{r})\right]dS\right\}$$

Discretization with HM:

BEM Discretization



$$\phi_g(\vec{r}) = \left(\underline{\phi}_g^R\right)^T \underline{h}^R(\vec{r}) + \left(\underline{\phi}_g^I\right)^T \underline{h}^I(\vec{r}), \ \vec{r} \in S^R$$

$$\phi_g(\vec{r}) = \left(\underline{\phi}_g^I\right)^T \underline{h}^I(\vec{r}), \ \vec{r} \in S^I$$

$$J_g(\vec{r}) = -\left(\underline{J}_g^I\right)^T \underline{h}^I(\vec{r}), \ \vec{r} \in S^I$$

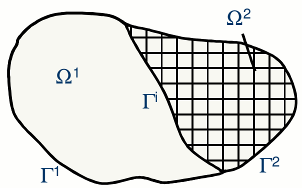
Reflector equations;

$$\underline{\underline{B}}_g^I \underline{\phi}_g^I + \underline{\underline{B}}_g^{IJ} \underline{J}_g^I + \underline{\underline{B}}_g^{IR} \underline{\phi}_g^R = \sum_{g'=1}^{g-1} s_{g,g'} \left[\underline{\underline{B}}_g^I \underline{\phi}_{g'}^I + \underline{\underline{B}}_g^{IJ} \underline{J}_{g'}^I + \underline{\underline{B}}_g^{IR} \underline{\phi}_{g'}^R\right]$$

$$\underline{\underline{B}}_g^R \underline{\phi}_g^R + \underline{\underline{B}}_g^{RI} \underline{\phi}_g^I + \underline{\underline{B}}_g^{RJ} \underline{J}_g^I = \sum_{g'=1}^{g-1} s_{g,g'} \left[\underline{\underline{B}}_g^R \underline{\phi}_{g'}^R + \underline{\underline{B}}_g^{RI} \underline{\phi}_{g'}^I + \underline{\underline{B}}_g^{RJ} \underline{J}_{g'}^I\right]$$

Discretization with HM:

HM Matrix Form - 1



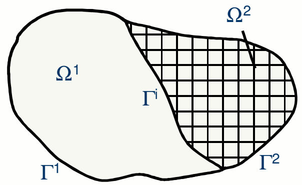
Arayüz ve sınır koşulları uygulanarak iki bölge denklem sistemleri birleştirilir:

$$\underline{\underline{M}}_g \underline{u}_g = \sum_{g'=1}^{g-1} \underline{\underline{S}}_{g\leftarrow g'} \underline{u}_{g'} + \frac{1}{k_{eff}} \sum_{g'=1}^{G} \underline{\underline{F}}_{g\leftarrow g'} \underline{u}_{g'}$$

$$\underline{\underline{M}}_g = \begin{bmatrix} \underline{\underline{A}}_g^{C} & \underline{\underline{A}}_g^{CI} & \underline{\underline{0}} & \underline{\underline{0}} \\ \left(\underline{\underline{A}}_g^{CI}\right)^T & \underline{\underline{A}}_g^{I} & \underline{\underline{A}}_g^{IJ} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{B}}_g^{I} & \underline{\underline{B}}_g^{IJ} & \underline{\underline{B}}_g^{IR} \\ \underline{\underline{0}} & \underline{\underline{B}}_g^{RI} & \underline{\underline{B}}_g^{RJ} & \underline{\underline{B}}_g^{R} \end{bmatrix}, \quad \underline{\underline{S}}_{g\leftarrow g'} = \begin{bmatrix} \underline{\underline{S}}_{g\leftarrow g'}^{C} & \underline{\underline{S}}_{g\leftarrow g'}^{CI} & \underline{\underline{0}} & \underline{\underline{0}} \\ \left(\underline{\underline{S}}_{g\leftarrow g'}^{CI}\right)^T & \underline{\underline{S}}_{g\leftarrow g'}^{I} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & s_{g,g'}\underline{\underline{B}}_g^{I} & s_{g,g'}\underline{\underline{B}}_g^{IJ} & s_{g,g'}\underline{\underline{B}}_g^{IR} \\ \underline{\underline{0}} & s_{g,g'}\underline{\underline{B}}_g^{RI} & s_{g,g'}\underline{\underline{B}}_g^{RJ} & s_{g,g'}\underline{\underline{B}}_g^{R} \end{bmatrix}$$

Discretization with HM:

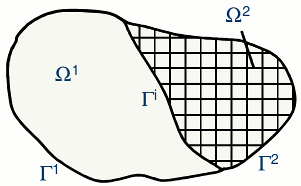
HM Matrix Form -2

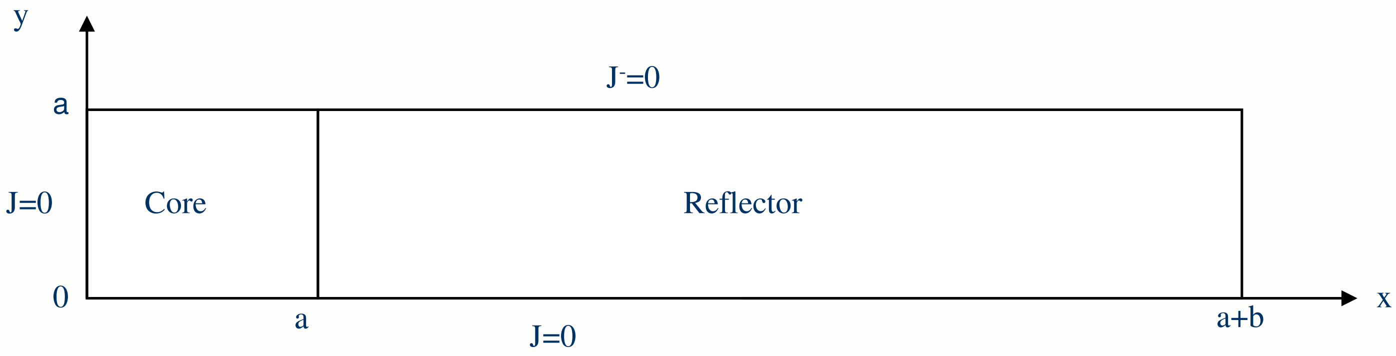


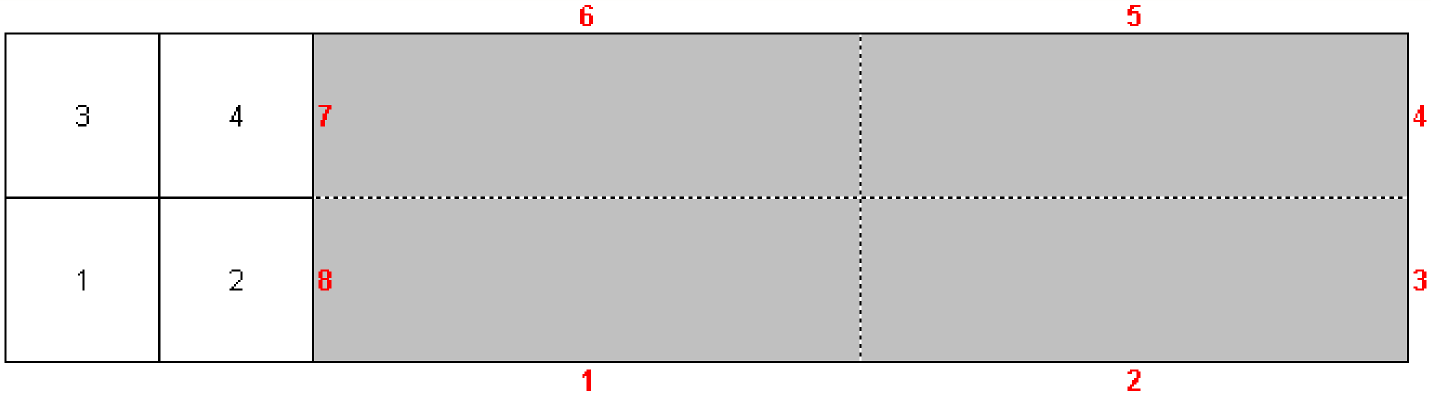
$$\underline{\underline{F}}_{g\leftarrow g'} = \begin{bmatrix} \underline{\underline{F}}^{C}_{g\leftarrow g'} & \underline{\underline{F}}^{CI}_{g\leftarrow g'} & \underline{\underline{0}} & \underline{\underline{0}} \\ \left(\underline{\underline{F}}^{CI}_{g\leftarrow g'}\right)^{T} & \underline{\underline{F}}^{I}_{g\leftarrow g'} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix}$$

$$\underline{u}_{g}^{T} = \left[\left(\underline{\phi}_{g}^{C}\right)^{T} \quad \left(\underline{\phi}_{g}^{I}\right)^{T} \quad \left(\underline{J}_{g}^{I}\right)^{T} \quad \left(\underline{\phi}_{g}^{R}\right)^{T} \right]$$

Applications







Application-1:

One Group – Fission Source

a = 22.5 cm, b = 20 cm

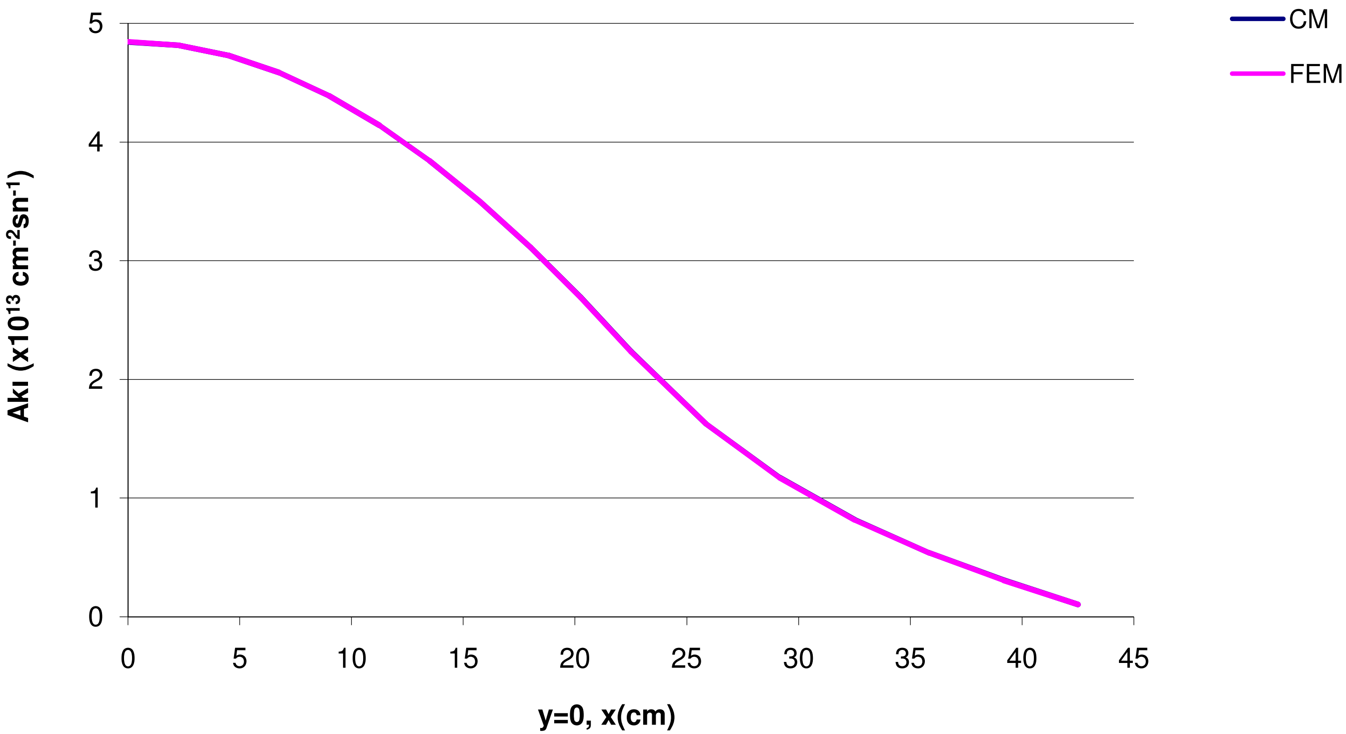
	Core	Reflector
$D_g(cm)$	0.87000	0.87000
$\Sigma_{r,g}(cm^{-1})$	0.01122	0.00330
$\upsilon\Sigma_{f,g}(cm^{-1})$	0.02305	0.00000

Analytic k_{eff} = 1.361959

k_{eff}					
Core	**Reflector**	**HM k_{eff}**	**HM Err**	**FEM k_{eff}**	**FEM Err**
3x5	3x3	1.35411	0.58%	1.35397	0.59%
5x5	5x3	1.35893	0.22%	1.35796	0.29%
6x10	6x6	1.36001	0.14%	1.35996	0.15%
10x10	10x6	1.36123	0.05%	1.36096	0.07%

Application-1:

One Group – Fission Source



Application-2:

Two Group – Fission Source

a = 4.86 cm, b = 20 cm

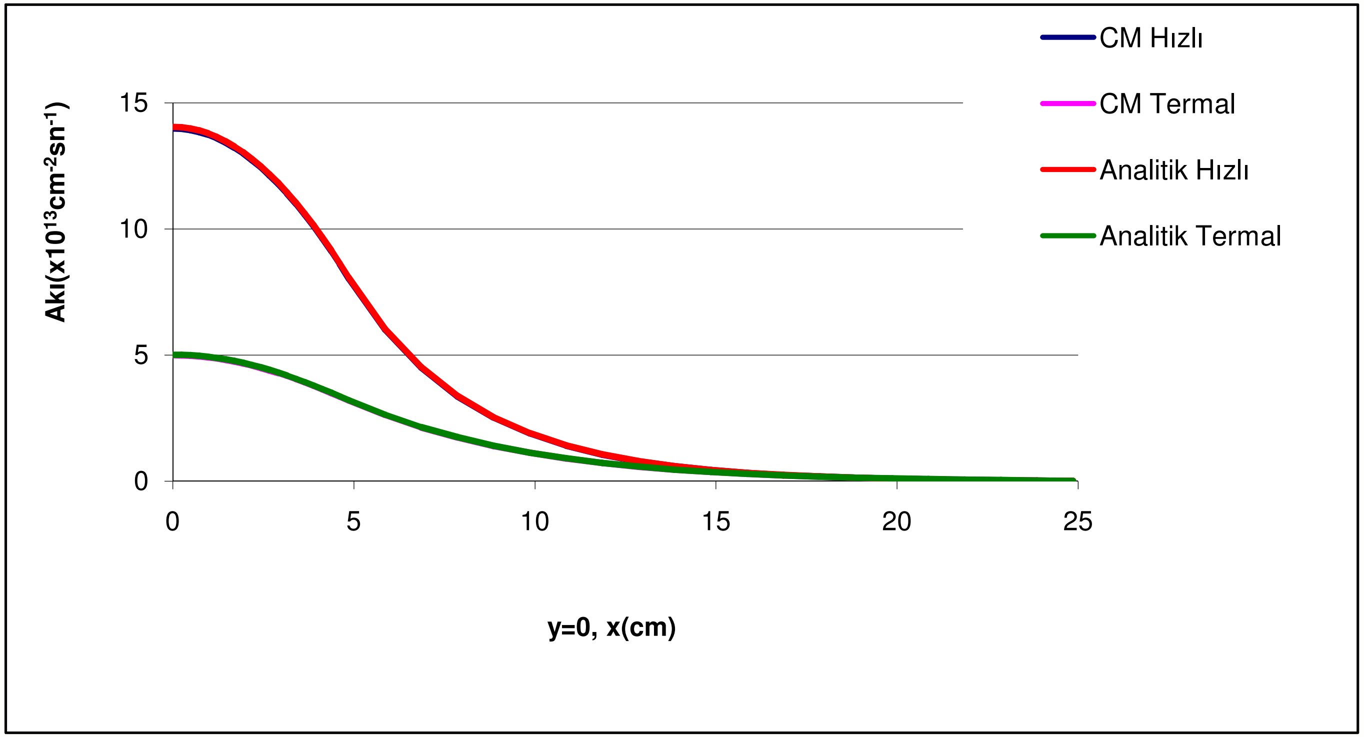
	Core		Reflector	
	I	II	I	II
$D_g(cm)$	0.6165356	0.6165356	0.6165356	0.6165356
$\Sigma_{r,g}(cm^{-1})$	0.080117	0.11484	0.01021	0.00267
$\upsilon\Sigma_{f,g}(cm^{-1})$	0.0813	0.17843	0.00000	0.00000
$\Sigma_{s,2\leftarrow 1g}(cm^{-1})$	0.063567	0.00000	0.01005	0.00000

Analytic k_{eff} = 1

k_{eff}					
Core	Reflector	HM k_{eff}	HM Err	CBEM k_{eff}	CBEM Err
4x5	4x5	0.99454	0.55%	0.97826	2.17%
8x10	8x10	0.99892	0.11%	0.99288	0.71%
16x20	16x20	0.99977	0.02%	0.99781	0.22%
32x40	32x40	-	-	0.99907	0.09%

Application-2:

Two Group – Fission Source



Application-3:

Three Group – Fission Source

a = 35 cm, b = 10 cm

	Group	Material	$D_g(cm)$	$\Sigma_{r,g}(cm^{-1})$	$\upsilon\Sigma_{f,g}(cm^{-1})$	$\Sigma_{s,g+1\leftarrow g}(cm^{-1})$	χ_g
CORE	1		166.400	0.04595000	0.00407154	0.04238949	1
	2		0.64800	0.09062400	0.00848000	0.06832400	0
	3		0.35120	0.12610000	0.18100000	0.0	0
REFLECTOR	1	Water	1.974	0.0733	0.0	0.07327	-
		Heavy Water	141.300	0.02759692	0.0	0.02758800	-
		Beryllium	0.83488	0.02007700	0.0	0.02007700	-
		Graphite	142.800	0.00728189	0.0	0.0072656	-
	2	Water	0.577	0.1501	0.0	0.1501	-
		Heavy Water	121.150	0.02404892	0.0	0.02404800	-
		Beryllium	0.47295	0.02101793	0.0	0.02099000	-
		Graphite	0.92816	0.00787369	0.0	0.00786740	-
	3	Water	0.16	0.0197	0.0	0.0	-
		Heavy Water	0.81641	0.00003324	0.0	0.0	-
		Beryllium	0.48616	0.00086727	0.0	0.0	-
		Graphite	0.86313	0.00020249	0.0	0.0	-

Application-3:

Three Group – Fission Source

FEM k_{eff} = 1.017036 (256x256 mesh)

	k_{eff}					
	Water			Heavy Water	Beryllium	Graphite
Mesh	Hybrid	FEM	BEM	Hybrid	Hybrid	Hybrid
16x16	1.016975 (-.0060%)	1.017591 (-.0556%)	1.017364 (.0323%)	1.024035	1.036552	1.024597
32x32	1.017017 (-.0019%)	1.017175 (.0137%)	1.017112 (.0075%)	1.024068	1.036583	1.024668
64x64	1.017029 (-.00069%)	1.017069 (.0032%)	1.017052 (.00160%)	1.024075	1.036591	1.024686

Conclusions & Suggestions

- HM converges to the analytic results as the number of meshes is increased.
- Results are consistent with that of pure FEM.
- Better error performance with respect to pure BEM.

Suggestions

- HM can be applied to multi-region diffusion equations with various geometries.
- Instead of linear elements, constant and quadratic elements can be used.
- Methods with better performance for the solution of linear systems can be employed.